

## 11-5 Integration using Trig Substitutions

### Learning Objectives:

Case #1:  $\sqrt{a^2 - x^2}$   Let  $x = a \sin \theta$

Case #2:  $\sqrt{a^2 + x^2}$   Let  $x = a \tan \theta$

Case #3:  $\sqrt{x^2 - a^2}$   Let  $x = a \sec \theta$

Ex1.  $\int x^3 \sqrt{16-x^2} dx$  Case #1  
 $x = 4 \sin \theta$   
 $\frac{dx}{d\theta} = 4 \cos \theta$   
 $dx = 4 \cos \theta d\theta$

$$\int (4 \sin \theta)^3 \sqrt{16 - (4 \sin \theta)^2} \cdot 4 \cos \theta d\theta$$

$$\int 64 \sin^3 \theta \sqrt{16 - 16 \sin^2 \theta} \cdot 4 \cos \theta d\theta$$

$$\int 64 \sin^3 \theta \sqrt{16(1 - \sin^2 \theta)} \cdot 4 \cos \theta d\theta$$

$$\int 64 \sin^3 \theta \sqrt{16 \cos^2 \theta} \cdot 4 \cos \theta d\theta$$

*perfect square*

$$\int 64 \sin^3 \theta \cdot 4 \cos \theta \cdot 4 \cos \theta d\theta$$

$$\int 1024 \sin^3 \theta \cos^2 \theta d\theta$$

$$1024 \int \sin^3 \theta \cos^2 \theta d\theta$$

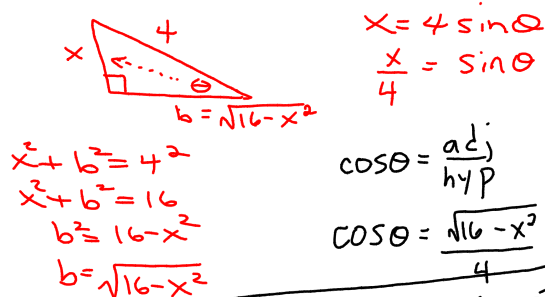
$$1024 \int \sin \theta \cos^2 \theta \sin^2 \theta d\theta$$

$$1024 \int \sin \theta \cos^2 \theta (1 - \cos^2 \theta) d\theta \quad \begin{matrix} u = \cos \theta \\ du = -\sin \theta d\theta \end{matrix}$$

$$-1024 \int u^2 (1 - u^2) du$$

$$-1024 \int u^2 - u^4 du = \frac{-1024 u^3}{3} + \frac{1024 u^5}{5} + C$$

$$\frac{-1024 \cos^3 \theta}{3} + \frac{1024 \cos^5 \theta}{5} + C$$



$$\frac{-1024}{3} \left[ \frac{\sqrt{16-x^2}}{4} \right]^3 + \frac{1024}{5} \left[ \frac{\sqrt{16-x^2}}{4} \right]^5 + C$$

$$\frac{-1024}{3} \frac{\sqrt{(16-x^2)}^3}{64} + \frac{1024}{5} \frac{\sqrt{(16-x^2)}^5}{1024} + C$$

$$\frac{-16}{3} \sqrt{(16-x^2)}^3 + \frac{1}{5} \sqrt{(16-x^2)}^5 + C$$

Ex2.  $\int \frac{dx}{\sqrt{x^2-9}}$   $\int \frac{dx}{\sqrt{x^2-3^2}}$

Case 3  $x = a \sec \theta$

$\int \frac{dx}{\sqrt{9 \sec^2 \theta - 9}} \left( \frac{3 \sec \theta \tan \theta d\theta}{\sqrt{9(\sec^2 \theta - 1)}} \right)$

$\int \frac{3 \sec \theta \tan \theta d\theta}{\sqrt{9 \tan^2 \theta}} \left( \frac{3 \sec \theta \tan \theta}{3 \tan \theta} \right) \int \sec \theta =$

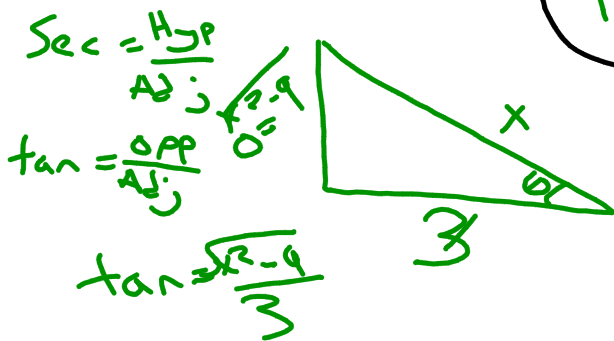
$x = 3 \sec \theta$   
 $\sec \theta = \frac{x}{3}$

$\sin^2 + \cos^2 = 1$   
 $\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$   
 $\tan^2 x + 1 = \sec^2 x$

$y = 3 \sec \theta$   
 $dx = 3 \sec \theta \tan \theta d\theta$

$\int \sec \theta =$   
 $\ln |\sec \theta + \tan \theta| + C$

$\ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| + C$



$x^2 = 3^2 + \text{opma}^2$   
 $\text{opma} = \sqrt{x^2-3^2}$

### Ex3. $\int \sqrt{1+x^2} dx$

$$\int \sqrt{1+x^2} dx \quad \begin{array}{l} x = \tan \theta \\ dx = \sec^2 \theta d\theta \end{array}$$

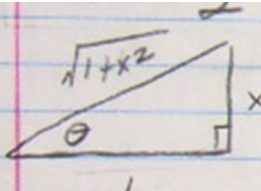
$$= \int \sqrt{1+(\tan \theta)^2} \cdot \sec^2 \theta d\theta$$

$$= \int \sqrt{1+\tan^2 \theta} \sec^2 \theta d\theta \quad 1+\tan^2 \theta = \sec^2 \theta$$

$$\int \sqrt{\sec^2 \theta} \sec^2 \theta d\theta$$

$$\int \sec \theta \cdot \sec^2 \theta d\theta = \int \sec^3 \theta d\theta$$

$$\frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$$



$$\sec \theta = \frac{\sqrt{x^2+1}}{1} \quad \tan \theta = \frac{x}{1}$$

$$\frac{1}{2} \frac{\sqrt{x^2+1}}{1} \cdot \frac{x}{1} + \frac{1}{2} \ln |\sqrt{x^2+1} + x| + C$$

$$\boxed{\frac{x\sqrt{x^2+1}}{2} + \frac{1}{2} \ln |\sqrt{x^2+1} + x| + C}$$

$$\text{Ex4. } \int \frac{1}{(1+x^2)^2} dx$$

Case # 2

$$x = \tan \theta$$

$$x = \tan \theta \quad \int \frac{1}{(1+\tan^2 \theta)^2} \cdot \sec^2 \theta d\theta$$

$$\frac{dx}{d\theta} = \sec^2 \theta$$

$$\int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} = \int \frac{1 d\theta}{\sec^2 \theta} = \int \cos^2 \theta d\theta$$

$$\int \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta$$

$$\frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C$$

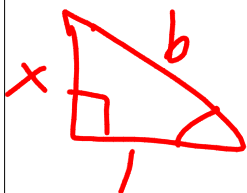
$$\tan \theta = \frac{x}{1}$$

$$b^2 = x^2 + 1$$

$$b = \sqrt{x^2 + 1}$$

$$\frac{1}{2} \tan^{-1} x + \frac{1}{2} \left( \frac{x}{\sqrt{x^2+1}} \right) \left( \frac{1}{\sqrt{x^2+1}} \right) + C$$

$$\boxed{\frac{1}{2} \tan^{-1} x + \frac{x}{2x^2+2} + C}$$



# Homework

Integrating Powers of Trig Functions  
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